6.1 - Review of Power Series

A power series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

In our work, $a=0$

so $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots = a_1 + a_2 + \sum_{i=3}^{\infty} a_i$

Definition: A function is **analytic at a point a** if it can be represented with a power series centered at a with radius $R \neq 0$.

Example: Rewrite the given expression using a single power series whose general term involves x^k.

① Make exponents match
② Make indices match

$$\sum_{n=1}^{\infty} nc_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2}$$

① Make exponents match

$k=n-1$ $\quad$ $k=n+2$
$n=k+1$ $\quad$ $n=k-2$

$$\sum_{k=0}^{\infty} (k+1)c_{k+1}x^k + \sum_{k=2}^{\infty} 3c_{k-2}x^k$$

$n=1 \quad n=2 \quad n=3$

$c_1 + 2c_2x + 3c_3x^2 \qquad 3c_0x^2 + 3c_1x^3$

$c_1 + 2c_2x$ etc $\qquad 3c_0x^2 + 3c_1x^3$

$$\boxed{c_1 + 2c_2x + \sum_{k=2}^{\infty}\left[(k+1)c_{k+1} + 3c_{k-2}\right]x^k}$$

$k=0 \quad k=1$ $\qquad$ $k=2: (3c_3 + 3c_0)x^2$

Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the given linear first-order differential equation.

$$y = ce^{-1/4\,x}$$

$4y' + y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

when we differentiate, the starting value increases by 1.

$$\sum_{n=1}^{\infty} 4 n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

① Exponents

cosmetic change

$$\sum_{k=0}^{\infty} 4(k+1) c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left[4(k+1) c_{k+1} + c_k\right] x^k = 0$$

$$4(k+1)\boxed{c_{k+1}} + c_k = 0 \quad \text{for all } k = 0, 1, 2, \ldots$$

solve for the constant with the largest subscript

$$\boxed{c_{k+1} = -\frac{1}{4(k+1)} c_k, \quad k = 0, 1, 2, \ldots}$$

recurrence relation

$$c_{k+1} = -\frac{1}{4(k+1)} c_k, \quad k = 0, 1, 2 \ldots$$

$k=0$: $c_1 = -\frac{1}{4} c_0$

$k=1$: $c_2 = -\frac{1}{4(2)} c_1 = -\frac{1}{4 \cdot 2}\left(-\frac{1}{4} c_0\right) = \frac{1}{2 \cdot 4^2} c_0$

$k=2$: $c_3 = -\frac{1}{4(3)} c_2 = -\frac{1}{3 \cdot 4}\left(\frac{1}{2 \cdot 4^2} c_0\right) = -\frac{1}{2 \cdot 3 \cdot 4^3} c_0$

$k=3$: $c_4 = -\frac{1}{4(4)} c_3 = -\frac{1}{4 \cdot 4}\left(-\frac{1}{2 \cdot 3 \cdot 4^3} c_0\right) = \frac{1}{4! \, 4^4} c_0$

$$c_n = (-1)^n \frac{1}{n! \, 4^n} c_0$$

Recall: $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

$$y = c_0 - \frac{1}{4} c_0 x + \frac{1}{2 \cdot 4^2} c_0 x^2 - \frac{1}{3! \cdot 4^3} c_0 x^3 + \cdots$$

$$y = c_0 \left(1 - \frac{1}{4} x + \frac{1}{32} x^2 - \frac{1}{384} x^3 + \cdots\right)$$

Most solutions in this chapter will look like this

But here we have $y = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \, 4^n} x^n$,

Which is $y = c_0 e^{-\frac{1}{4} x}$, as expected.

Example: The given function is analytic at $a = 0$. Use appropriate series and multiplication to find the first four nonzero terms of the Maclaurin series of the given function.

$e^{-x} \cos x$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$e^{-x}\cos x = \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\right)$$

$$\begin{aligned}
= \; & 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \\
& - x + \frac{1}{2}x^3 - \frac{1}{24}x^5 \\
& + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{24}x^6 \\
& - \frac{1}{6}x^3 + \frac{1}{12}x^5 - \frac{1}{6\cdot 24}x^7 \\
& + \frac{1}{24}x^4 + \cdots
\end{aligned}$$

$$e^{-x}\cos x = 1 - x + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$$